Quanta of Knowledge

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This question carries tremendous fame (or massive notoriety!) at its back – "the most difficult question in the history of IIT JEE", "a particularly challenging problem owing to asymmetric mass distribution and dual rotation", "the hardest problem ever". This "legendary problem" is often cited as an example to highlight the toughness level of JEE.

But the truth is, this is a very easy question. This question is just about careful calculations. All that this question requires is some simple geometry and trigonometry, very simple vector operations - resolution and cross-product, and absolutely elementary concepts of rotational motion. It has ridiculously simple solutions.

Here are my experiences of interactions with JEE aspirants and teachers on this question. What follows is a general discussion around the question; only four-five lines and one diagram constitute the solution. There are some issues with the framing of the question. I will touch upon those issues also.

Two thin circular discs of mass m and $4m$, having radii of a and $2a$, respectively, are rigidly fixed by a massless, rigid rod of length $l = \sqrt{24}a$ through their centres. This assembly is laid on a firm and flat surface, and set rolling without slipping on the surface so that the angular speed about the axis of the rod is ω. The angular momentum of the entire assembly about the point 'O' is $\vec{L}$ (see the figure). Which of the following statement(s) is(are) true?

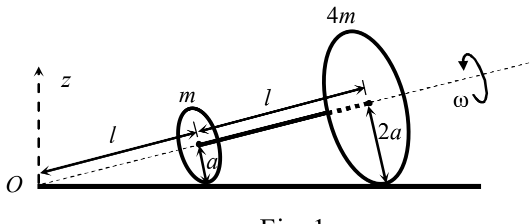


Fig. 1

(A) The centre of mass of the assembly rotates about the z- axis with an angular speed of $\frac{\omega}{5}$.

(B) The magnitude of angular momentum of centre of mass of the assembly about the point O is $81\, ma^2\omega$.

(C) The magnitude of angular momentum of the assembly about its centre of mass is $\frac{17}{2} ma^2\omega$.

(D) The magnitude of the z- component of $\vec{L}$ is $55\, ma^2\omega$.

JEE (Advanced – 2016, 4 Marks)

You can solve this problem by using the fundamental theorem of angular momentum - $\vec{L} = \vec{L}_{\text{motion of cm}} + \vec{L}_{\text{about cm}}$, and $L = I\omega$ (you know when to use it when not to), along with equations $v = r\omega$, $\vec{\omega} \times \vec{r} = \vec{v}$ etc. It should not take more than 7-8 minutes; physics requires 1 minute, calculations 6-7 minutes.
Or, you can decorate the problem with an exotic technical jargon and make it look extremely complicated and difficult. I wonder why students introduce terms like asymmetric mass distribution (whereas mass distribution is highly symmetrical - that of a uniform disc about its axis just set the line of your sight cleverly), dual rotation (so what, we have remarkably elegant addition (superposition) techniques), translatory frame, rotatory frame, axis of symmetry, principal axes of rotation, moment of inertia tensors I_{xx}, I_{yy}, I_{zz}, I_{xy}..., spin angular momentum, orbital angular momentum etc. in solving this problem. (These terms suit to Ferdinand Beer and Russell Johnston or the legendary PC Dumir (IIT Delhi graduates may remember him).) These heavy-duty words create an impression that the problem is of advanced nature, beyond the JEE syllabus. No, it's not.

First, we shall dissect the system and its motion. (*We do not need it for the solution. Let us do it nonetheless. It is important to understand the forces at play in the problem by dividing it into parts. Making a mental picture of the problem is the first step of problem solving.)* Assume axes *x*- and *y*- in the firm flat (horizontal) surface. The vertical *z*- axis passes through their intersection *O*. The *y*- axis is shown by ⊗ in the figure. This direction comes from $\hat{i} \times \hat{j} = \hat{k}$. *Figure depicts the arrangement at the instant* the instantaneous points of contact of the discs (points P_1 and P_2) lie on *x*- axis. The axis Ox' is in *x*-*z* plane, at an angle of θ with the *x*- axis. As seen from *O*, the assembly rotates about line Ox' in anticlockwise sense with angular speed ω. The assembly is rolling on the horizontal surface without slipping. Suppose the centre of mass of the assembly rotates about *z*- axis with angular speed ω_z. This is essentially the speed of the rotation of massless, light rod rigidly fixed to the centres of the discs (or rotation of line Ox') about *z*- axis in anticlockwise sense as seen from a (distant) point on positive *z*- axis while looking towards *O*..

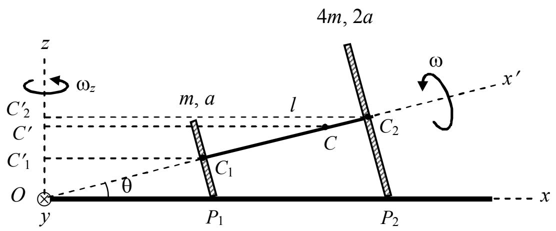


Fig. 2

Let us denote by C_1, C_2 and C the centres of mass of the small disc, big disc and the whole assembly, respectively. We see that $\angle OC_1P_1 = 90^\circ$, (axis is normal to the plane of the disc) $OP_1 = \sqrt{(OC_1)^2 + (C_1P_1)^2} = \sqrt{l^2 + a^2} = \sqrt{24a^2 + a^2} = 5a$, $\sin\theta = \frac{1}{5}$, $\cos\theta = \frac{\sqrt{24}}{5}$.

Distance of centre of mass of the assembly from O is $OC = \frac{m \times l + 4m \times 2l}{5m} = \frac{9l}{5}$, whence $C_1C = \frac{4l}{5}$, $CC_2 = \frac{l}{5}$.

The centre of mass C_1 of the small disc moves in a circle in a horizontal plane: centre - C_1', radius $= C_1'C_1 = l\cos\theta$, speed $= (C_1'C_1)\omega_z = (l\cos\theta)\omega_z$; C moves in a circle in a different horizontal plane (a little above the plane of motion of C_1): centre - C', radius $= C'C = \frac{9l}{5}\cos\theta$, speed $= (C'C)\omega_z = \left(\frac{9l}{5}\cos\theta\right)\omega_z$; C_2 moves in a circle in another horizontal plane: centre - C_2', radius $= C_2'C_2 = 2l\cos\theta$, speed $= (C_2'C_2)\omega_z = (2l\cos\theta)\omega_z$.

In vector notation, velocity of C_1 is $(l\cos\theta)\,\omega_z\hat{j}$, velocity of C is $\left(\frac{9l}{5}\cos\theta\right)\omega_z\hat{j}$, velocity of C_2 is $(2l\cos\theta)\,\omega_z\hat{j}$ at the instant the assembly is on the x- axis we have drawn on the flat surface.

Velocities of these points, or for that matter, of any point on the axis of the assembly, are always horizontal. This doesn't mean that they move with constant velocities. Their velocities turn about z- axis.

The motion of the assembly is a combination of two plane rotations - plane rotation with angular speed ω about its own axis (in anticlockwise sense as seen from O) and plane rotation with angular speed ω_z about z- axis (in anticlockwise sense as seen from top). We combine the effects of these two plane rotations (Fig. 3) for whatever we want to compute.

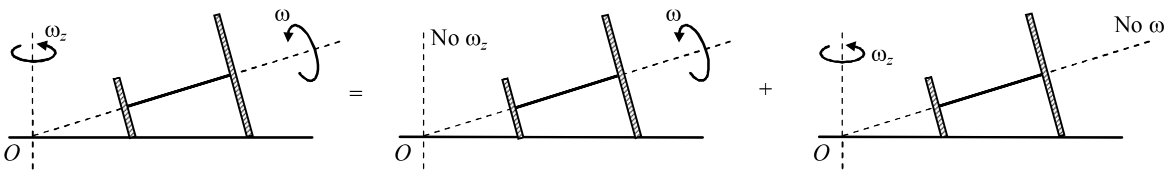


Fig. 3

We shall now check if statement **(A)** is true. Let us look at the instantaneous point of contact P_1 (Fig. 2). Plane rotation about Ox' gives it a velocity $-a\omega\hat{j}$ and plane rotation about z- axis gives it a velocity $(5a)\omega_z\hat{j}$. (In plane rotation about z- axis, point P_1 moves in a circle of radius $5a$ with angular speed ω_z.) Since the instantaneous velocity of point P_1 is zero

$$(5a)\omega_z\hat{j} + (-a\omega j) = 0$$

which gives $\omega_z = \frac{\omega}{5}$.

(One should not have any difficulty in figuring out the velocities of P_1 due to the two rotations. If the tilting of the disc is troubling you, look at the velocity of P_1 in the three cases in Fig. 4, which shows a vertical, a tilted and a horizontal disc.

In all the three cases, velocity of P_1 is $a\omega$ along $-y$ direction, shown by ⊙ in the figure.)

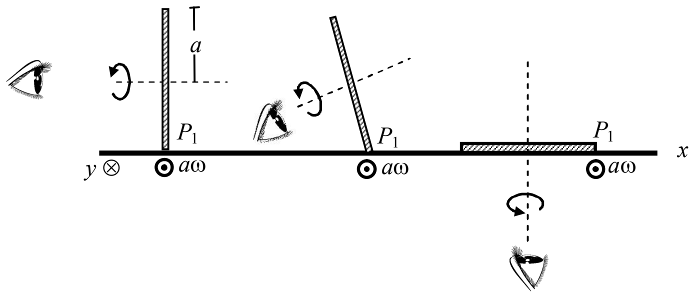
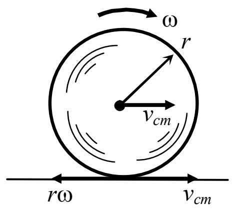


Fig. 4

This is almost similar to the way we relate the two velocities for a disc rolling on a horizontal surface, $v_{cm} = r\omega$. What is the difference between this motion and the one given in the problem?



In exam, a student just needs to write: $(5a)\omega_z = a\omega$ or $\omega_z = \frac{\omega}{5}$.

(Statement **(A)** is true.)

We can apply many lateral ideas to arrive at this result. Let us watch the motion of the smaller disc. In one complete rotation, each point on its circumference comes in contact with a point on the horizontal circle of radius $5a$. The circumference $2\pi a$ is "carpeted" over an arc of length $2\pi a$ (shown as arc $P_1 P_1'$ in Fig. 5.)

Angular speed of its centre of mass C_1 about z- axis is

$$\frac{\phi}{t} = \frac{\left(\frac{\widehat{PP'}}{5a}\right)}{t} = \frac{\left(\frac{2\pi a}{5a}\right)}{\left(\frac{2\pi}{\omega}\right)} = \frac{\omega}{5}.$$

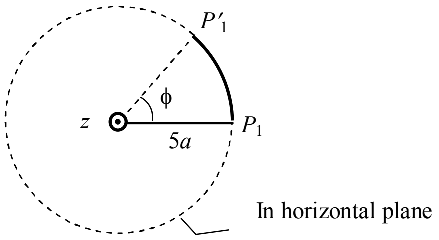
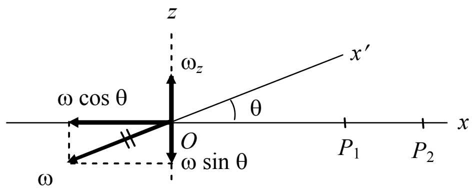


Fig. 5

Alternatively, the two radii a and $5a$ suggest that for the complete "carpeting" of the bigger circle once the disc must turn about its axis 5 times, $2\pi(5a) = 5\times(2\pi a)$. Rotation of the disc's axis about z- axis must be 5 times slower than rotation of the disc about its own axis. This gives $\omega_z = \frac{\omega}{5}$.

Finally, a little mathematical treatment. You know that points P_1 and P_2 on the assembly are momentarily at rest. Line OP_1P_2 is instantaneous axis of rotation. The angular velocity of the assembly about O, whatever it is, it must entirely be along this axis. And for that to happen, $\omega_z = \omega\sin\theta = \frac{\omega}{5}$, Fig. 6. (*We see that angular speed of the assembly about instantaneous axis of rotation is* $\omega\cos\theta$. *Using this information we can answer options,* ***(C)*** *and* ***(D)*** *straightaway.*)



Instantaneous points of contacts are shown by points P_1 and P_2 marked on x- axis

Fig. 6

To fine tune your vector algebra, you can also try the following: $\vec{v}_P = \vec{\omega}\times\vec{r}$ holds for any point P on the body, provided $\vec{r}$ is the position vector of point P from any point on the instantaneous axis.

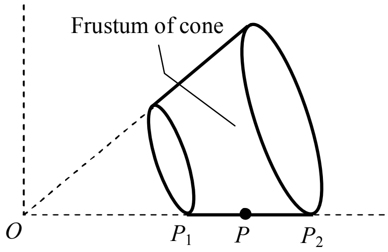
$$\vec{v}_{P_1} = 0 \Rightarrow (\vec{\omega}_z + \vec{\omega})\times\overrightarrow{OP_1} = 0,$$

and $$\vec{v}_{P_2} = 0 \Rightarrow (\vec{\omega}_z + \vec{\omega})\times\overrightarrow{OP_2} = 0.$$

From any one of the above equations, you can get $\omega_z = \frac{\omega}{5}$.

In fact, you can visualize the system as a frustum of a cone with the portion between the two discs being massless and invisible.

Since the points P_1, P, P_2 etc., lying on the x- axis and on the frustum have zero velocity, we have



$$(\vec{\omega}_z + \vec{\omega}) \times \overrightarrow{OP} = 0$$

or $$\left[\omega_z \hat{k} + (-\omega\cos\hat{i} - \omega\sin\theta\hat{k})\right] \times x\hat{i} = 0, \text{ where } x = \left|\overrightarrow{OP}\right|,$$

or $$x(\omega_z - \omega\sin\theta)\hat{k} = 0.$$

This gives

$$\omega_z = \omega\sin\theta = \frac{\omega}{5}.$$

Now we shall look at Statement **(B):** This statement is a little interesting. We understand the notion of a particle, a collection of particles, a body, a system of bodies. We know the concept of centre of mass. We absolutely clearly understand what the centre of mass of a system is, how to find its position, and how to describe its motion. Let us first refine the concept of centre of mass with the help of a simple example. Consider a thin uniform ring made of copper. The mass of the ring is m and its radius R. The temperature of the ring is T. The ring is rotating in anticlockwise sense about its axis (a line through its centre of mass (*CM*) and normal to its plane) with an angular speed ω. Try to answer the following questions:

(i) What is colour of *CM*?
(ii) What is temperature of *CM*?
(iii) What is price of *CM*?
(iv) What is wavelength of the *CM*? Its focal length?
(v) Is $mR^2\omega^2$ the kinetic energy of the ring or of its *CM*?
(vi) Suppose ω increases at a constant rate of $\frac{d\omega}{dt}$. Is $2mR^2\omega\frac{d\omega}{dt}$ power of *CM* of the ring or power supplied to the ring by the external agent who is increasing ω?
(vii) What was All India Rank of *CM* in JEE (Advanced) - 2016?

The centre of mass of a system of particles is a *point* which moves *AS IF* the whole mass of the system were concentrated at that point. We speak of position vector $\vec{r}_{cm}$ $\left(\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i}\right)$, velocity $\vec{v}_{cm}$ $\left(\vec{v}_{cm} = \frac{\sum m_i \vec{v}_i}{\sum m_i}\right)$ and acceleration $\vec{a}_{cm}$ $\left(\vec{a}_{cm} = \frac{\sum m_i \vec{a}_i}{\sum m_i}\right)$ of the centre of mass. Do we ever speak of kinetic energy of the centre of mass? Or its potential energy? Have you ever written $K_{cm} = \frac{\sum m_i K_i}{\sum m_i}$? Do you ever write $U_{cm} = \frac{\sum m_i U_i}{\sum m_i}$? Have you ever seen the equation $\vec{L}_{cm} = \frac{\sum m_i \vec{L}_i}{\sum m_i}$?

God knows what is the magnitude of angular momentum *of* centre of mass of the assembly about point O!

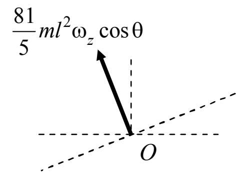
Angular momentum about point O attributable to the motion of centre of mass (or simply, of motion of centre of mass) of the assembly is

$$\vec{L}_{O,\text{ motion of } cm} = 5m \times \frac{9l}{5} \times \left(\frac{9l}{5}\cos\theta\right)\omega_z \qquad \ldots\text{(i)}$$

$$= \frac{81m}{5} l^2 \omega_z \cos\theta = \frac{81}{5} m (5a\cos\theta)^2 (\omega\sin\theta)\cos\theta \qquad \left(\cos\theta = \frac{l}{5a}\right)$$

$$= 81ma^2\omega\cos^3\theta \qquad \left(\sin\theta = \frac{1}{5}\right)$$

$$\neq 81ma^2\omega.$$

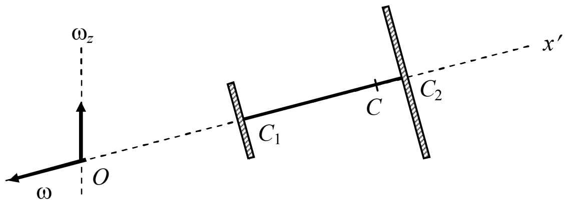


Statement **(B)** is not true.

*(We have, $\cos\theta = \frac{\sqrt{24}}{5} \approx 0.98$, $\cos^3\theta = 0.94$. Hence, $L_{\text{motion of } cm} \approx 81ma^2\omega$. Statement **(B)** can be taken to be true. It is not a bad approximation per se. But the examiner didn't like it.)*

We shall now look at Statement **(C)** and **(D)**. There are many routes that lead us to the answer. Half a dozen of them. All are very beautiful, pleasant and lovely. Application of the fundamental theorem of angular momentum makes it child's play. We shall calculate the angular momenta due to the two plane rotations separately and then add them. Superposition principles allow this.

You do not need the firm flat surface, the massless rigid rod, the *x*-, *y*- axes, point P_1 and P_2, rolling without slipping condition anymore. The Arjun in you should see only the eye of the fish. Keep your eyes at point *O*, look at the discs, *only discs*, one behind the other, rotating and moving, their motion characterized by the angular velocities ω and ω_z.



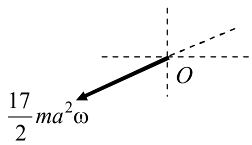
The angular momentum due to the two rotations can be written straightaway. I will calculate the magnitudes and show the directions diagrammatically. This helps.

Angular momentum due to plane rotation about axis *Ox′*:

$$L_{\text{smaller disc}} = \frac{1}{2}ma^2\omega$$

$$L_{\text{bigger disc}} = \frac{1}{2}4m(2a^2)\omega.$$

Hence, $L_{\text{for both discs}} = \frac{17}{2}ma^2\omega.$



We could have directly written:

$$L = \left(\frac{1}{2}ma^2 + \frac{1}{2}4m(2a)^2\right)\omega = \frac{17}{2}ma^2\omega.$$

Angular momentum contributed by plane rotation with angular speed ω_z about *z*-axis:

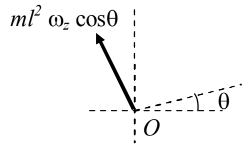
We will use the theorem,

$$\vec{L} = \vec{L}_{\text{montion of } cm} + \vec{L}_{\text{about } cm} = \vec{R}_{cm} \times M\,\vec{v}_{cm} + \vec{L}_{\text{about } cm}.$$

We can apply this formula to the assembly as a whole. (We will do it later, this makes a good exercise.) There is a far better way out. Calculate angular momentum for the two discs separately and then add them. You can be even smarter; calculate it for the small disc and then replace ma^2 by

$ma^2 + 4m(2a)^2 = 17ma^2$ and ml^2 by $ml^2 + 4m(2l)^2 = 17ml^2$ in the expression you obtain. You will get L for the whole assembly. What a beautiful symmetry!!

For the small disc



$$L_{\text{motion of } cm} = m \times l \times (\omega_z\, l\cos\theta) = ml^2\omega_z \cos\theta$$

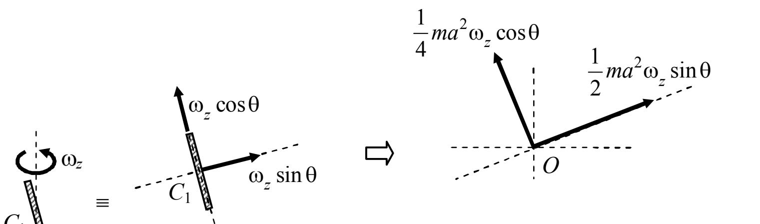
For $\vec{L}_{\text{about}\, cm}$ we need motion of the disc about its *CM*. It rotates with ω_z about the vertical line through the CM. How? Can you see through this? It's easy. Distance of C_1 from *z*- axis is $l\cos\theta$. The tilted disc rotates about *z*- axis with angular speed ω_z. We need its motion about its CM. How will the motion variables change if the distance of C_1 from *z*- axis were $1.2l\cos\theta$ or $5l\cos\theta$ or 100 metres or 1000 kilometers? What if this distance were zero? Distance of C_1 from *z*- axis is zero and the disc is rotating about *z*- axis with angular speed ω_z! Thus, for an observer at C_1, the disc rotates about vertical axis through C_1 with angular speed ω_z.

And this brings us to a very familiar motion - rotation of a disc about one of its diameters and about its axis simultaneously. In such motions we add the two ω's to get the resultant ω, and add the corresponding angular momenta for total angular momentum. What can be added can be split also. We know how to find angular momentum in simple cases. Here, we can divide to conquer, and then bunch the conquests up.

Split ω_z along the axis and the 'friendly' diameter, find the corresponding angular momenta and add them. (A word of advice; don't write $L = I\omega_z$ blindly, even if you know that $I = \frac{1}{4}ma^2(1+\sin^2\theta)$.* You must be very careful in using $L = I\omega$. NCERT physics textbook explains very clearly what you can do, what you cannot do. For those who are still not convinced about these arguments, the following should suffice-

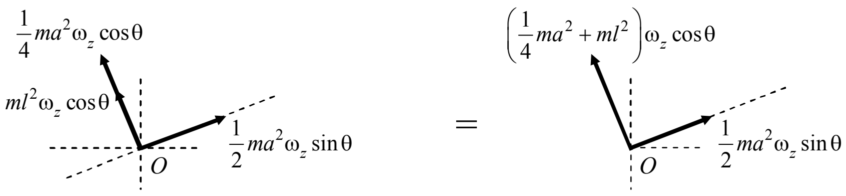
$$\begin{aligned}
\vec{L} &= \sum m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \\
&= \sum m_i \vec{r}_i \times [(\vec{\omega}_{\text{along axis}} + \vec{\omega}_{\text{along diameter}}) \times \vec{r}_i] \\
&= \sum m_i \vec{r}_i \times (\vec{\omega}_{\text{along axis}} \times \vec{r}_i) + \sum m_i \vec{r}_i \times (\vec{\omega}_{\text{along diameter}} \times \vec{r}_i) \\
&= I_{axis}\, \vec{\omega}_{\text{along axis}} + I_{\text{diameter}} \vec{\omega}_{\text{along diameter}}.
\end{aligned}$$

You must have derived these simple equations. You know them. In case these simple equations are stranger to you, derive them first.)



$\vec{L}_{\text{about}\, cm}$ of small disc due to ω_z

The theorem $\vec{L} = \vec{L}_{\text{motion of}\, cm} + \vec{L}_{\text{about}\, cm}$ gives:



$\vec{L}_O$ of small disc due to both ω and ω_z :

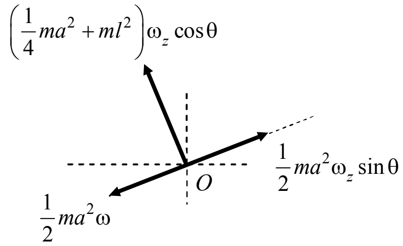


Fig. 7

Finally, $\vec{L}_O$ of the whole assembly:

(Replace ma^2 by $17ma^2$ and $ml^2=17ml^2$) (Note that $\omega_z=\omega\sin\theta$),

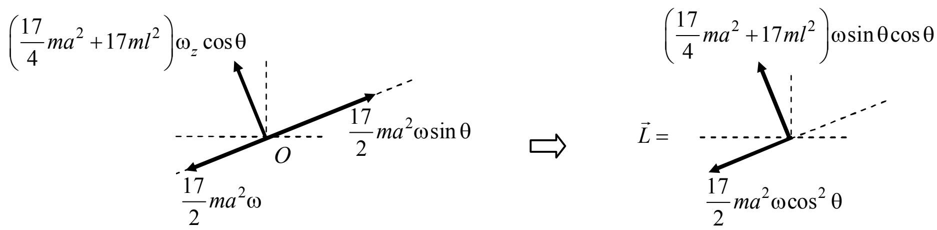
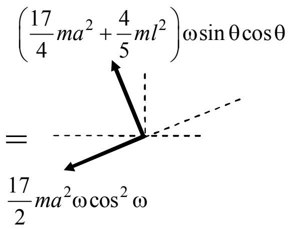


Fig. 8

Now, we can also calculate $\vec{L}_{\text{about } cm}$ as follows:

$\vec{L}_{\text{about } cm}=\vec{L}_O-\vec{L}_{\text{motion of } cm}$ =



How did we get $\frac{4}{5}$? I hope you understand.

$\left(17-\frac{81}{5}=\frac{4}{5}\right)$

Fig. 9

You can show by calculations that $L_{\text{about } cm}\neq\frac{17}{2}ma^2\omega$. (We see that $\frac{17}{2}ma^2\omega\cos^2\theta\approx\frac{17}{2}ma^2\omega$, though not exact. But there is another component.)

And, z- component of angular momentum of the assembly,

$$L_z=\left(\frac{17}{4}ma^2+17ml^2\right)\omega\cos^2\theta\sin\theta-\frac{17}{2}ma^2\omega\cos^2\theta\sin\theta$$

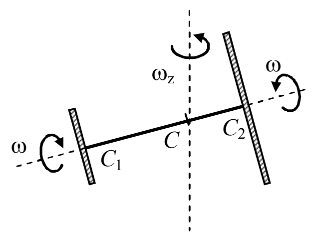
$$=\left(17ml^2-\frac{17}{4}ma^2\right)\omega\cos^2\theta\sin\theta=17ma^2\omega\left(24-\frac{1}{4}\right)\cos^2\theta\sin\theta\neq 55ma^2\omega\,.$$

So, statements **(C)** and **(D)** are not true.

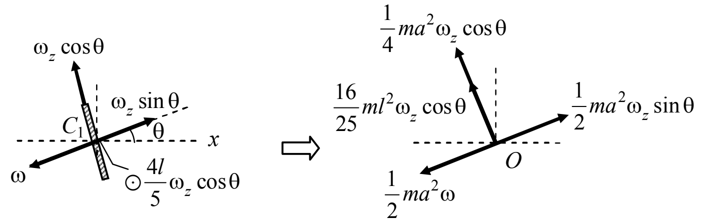
(What a cruelty! The examiner could have been a bit considerate. Had he given what I calculated, he wouldn't have lost anything; I am losing my confidence, my composure and my precious exam time. I will falter in other questions too.)

In examiner's view statement **(C)** is true. Strange!! What may be true (statement **(B)**) is true not, what is not at all true (Statement **(C)**) is said to be true. Saraswati naH subhaga mayskaraT.
I tried to guess what mistakes should one commit to get the expressions given in statements **(C)** and **(D)** as the answer. I even tried inappropriate approximations. I didn't succeed. Can someone help?
(We decided to compute the angular momentum of the small disc about point O.** Contribution of ω can be written directly. For the contribution of ω_z, we first calculated $\vec{L}_{\text{motion of } cm}$, the angular momentum due to motion of C_1 about O, and then $\vec{L}_{\text{about } cm}$, which is L of the disc about its centre of mass. Then we added them all. ***You can draw all contributions due to all rotations in a single diagram. Try it. It's one - minute job.***)

Now an interesting exercise. We shall look at the assembly from point O. We apply the fundamental theorem and compute the angular momentum $\vec{L}_O$. Calculation of $\vec{L}_{\text{motion of } cm}$ is easy (Eq. (i)). Calculation of $\vec{L}_{\text{about } cm}$ seems difficult at first glance. It is not difficult at all. We can apply the fundamental theorem once again. We look at the discs from point C, calculate $\vec{L}_{\text{motion of } C_1}$ and $\vec{L}_{\text{about } C_1}$ (for the small disc) and $\vec{L}_{\text{motion of } C_2}$ and $\vec{L}_{\text{about } C_2}$ (for the big disc.) Adding these $\vec{L}$'s we get $\vec{L}_{\text{about } cm}$ for the assembly as a whole. Successive application of the fundamental theorem simplifies the solution a great deal. We can write all $\vec{L}$'s directly. Then add them carefully. Let us see how. What is the motion of assembly like as seen by an observer at C? Rather, what is the motion of the two discs when you look at them from C, the centre of mass of the assembly? Try to figure it out. The small disc rotates about CC_1 in clockwise sense (ω *directed away from C*) with angular speed ω. Velocity of C_1 relative to C is $-\frac{4l}{5}\omega_z\cos\theta\hat{j}$. This disc also rotates about a vertical axis through C_1 with angular speed ω_z.
How interesting! The bigger disc rotates in anticlockwise sense (ω *directed towards C*) about CC_2 with angular speed ω. Point C_2 has velocity $\frac{1}{5}l\omega_z\cos\theta\hat{j}$. The disc also rotates in anticlockwise sense with angular speed ω_z about a vertical axis through its centre of mass C_2.

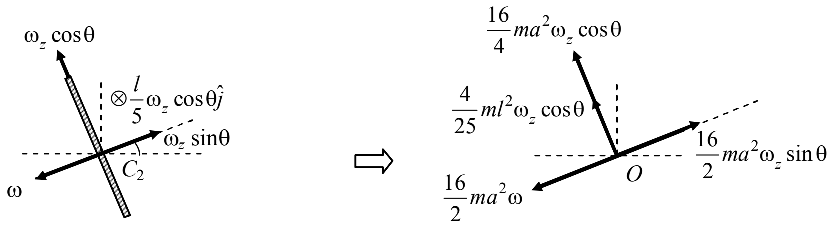


We can write the angular momentum of the discs due to these motions directly now.



Motion of small disc about C Angular momentum of small disc about C

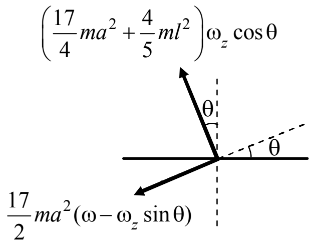
Fig. 10



Motion of bigger disc about C Angular momentum of bigger disc about C

Fig. 11

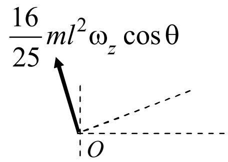
Combining L's in Figs. 10 and 11, gives the angular momentum of the assembly in the centre of mass of the system.



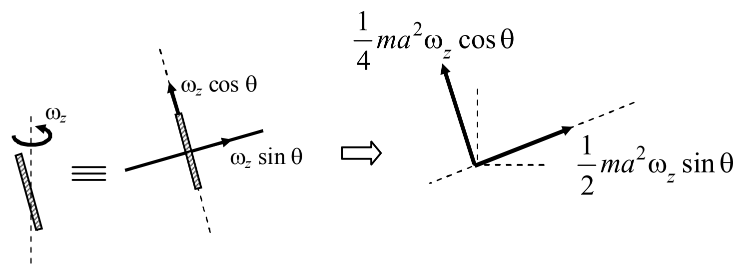
[By now you should not have any difficulty in writing L's due to different ω's. You know how to use the fundamental theorem. Yet, to remove even the slightest doubts, here are the calculation steps:

Angular momentum small disc about C (Using $\vec{L}_{\text{motion of } cm} = \vec{R}_{cm} \times M\,\vec{v}_{cm}$):

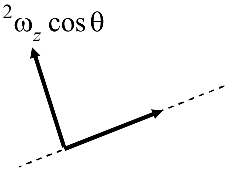
$$L_{\text{motion of } cm} = m \times \frac{4l}{5} \times \frac{4l}{5}\omega_z \cos\theta = \frac{16}{25}ml^2\omega_z \cos\theta$$



$L_{\text{about } cm}$:

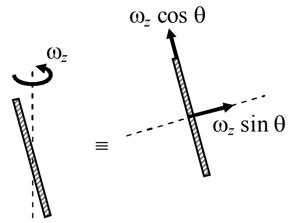


Angular momentum of bigger disc about *C*:



$$L_{\text{motion of } cm} = 4m\times\frac{l}{5}\times\frac{l}{5}\omega_z\cos\theta = \frac{4}{25}ml^2\omega_z\cos\theta$$

$L_{\text{about } cm}$



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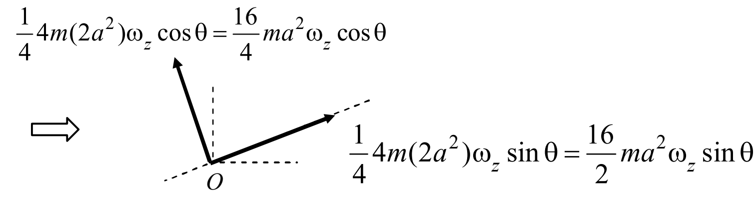
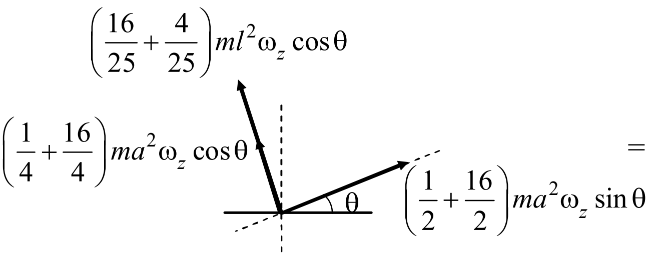


Now just combine all *L*'s:



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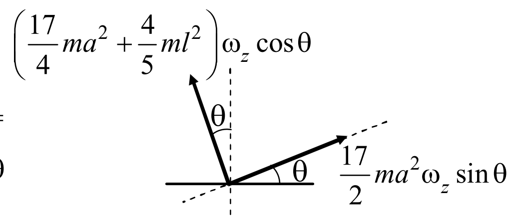
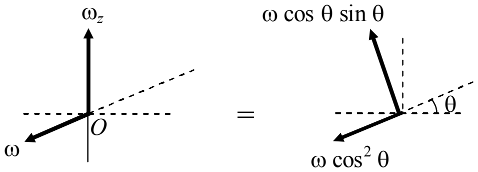


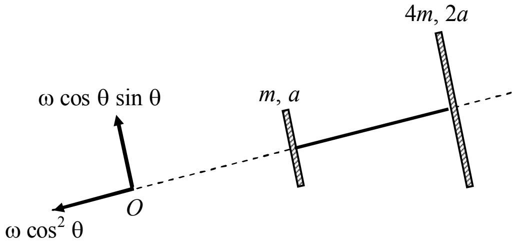
Caution: Unless you are very careful, you may commit a blunder at this point. After computing $\vec{L}_{\text{motion of } cm}$ for the two discs separately you may add the two expressions and claim to get $\vec{L}_{\text{motion of } cm}$ for the assembly. You must understand that $\vec{L}$'s are *not* added that way. We certainly have $m_1\vec{v}_{cm_1} + m_2\vec{v}_{cm_2} = (m_1+m_2)\vec{v}_{cm}$. This comes directly from the definition of centre of mass. But $\vec{r}_1\times m_1\vec{v}_{cm_1} + \vec{r}_2\times m_2\vec{v}_{cm_2} \neq \vec{r}_{cm}\times(m_1+m_2)\vec{v}_{cm}$. One can easily prove it.

We see that $L_{\text{about } cm}$ due to ω alone is $\frac{17}{2}ma^2\omega$. Plane rotation about *z*- axis with angular speed ω_z will also contribute to $L_{\text{about } cm}$. Contribution of ω_z can't be equal to zero. So the magnitude of angular momentum of the assembly about its centre of mass cannot be equal to $\frac{17}{2}ma^2\omega$. Part **(C)** is *not* true. Students at times argue that the examiner might have asked for "spin angular momentum". How does the word spin angular momentum fit in this scenario? "Spin angular momentum" is one thing, "angular momentum about centre of mass" is an altogether different thing. Moreover statement **(B)** is about "angular momentum of motion of centre of mass", and statement **(C)** about "angular momentum about the centre of mass". This clearly hints that the examiner wanted us to play with the equation $\vec{L} = \vec{L}_{\text{motion of } cm} + \vec{L}_{\text{about } cm}$.
It is unreasonable to read "angular momentum about centre of mass" as "spin angular momentum".

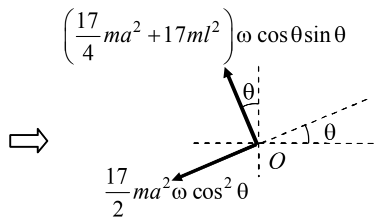
The above solution and calculations need clarity, and patience. An exam is about scoring, not trying out. Here is how to find L's almost in the stipulated time.

Just resolve ω_z along the axis Ox' and perpendicular to it. Angular velocity along the axis is $\omega - \omega_z \sin\theta = \omega - (\omega\sin\theta)\sin\theta = \omega - \omega\sin^2\theta = \omega\cos^2\theta$, and normal to the axis is $\omega_z\cos\theta = \omega\sin\theta\cos\theta$. You can arrive at these ω's in another way also. It was stated earlier (see Fig…) that the net angular velocity must be along the instantaneous axis of rotation. We found it to be $\omega\cos\theta$. Resolve $\omega\cos\theta$ in two components, component $(\omega\cos\theta)\cos\theta = \omega\cos^2\theta$ along the axis Ox' and the component $\omega\cos\theta\sin\theta$ perpendicular to it. Thus





Angular velocities along two convenient axes



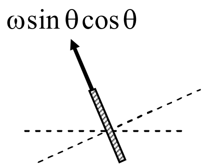
Resulting angular momenta

(Computation of angular momentum due to $\omega\cos^2\theta$ is straightforward. Just write the result.

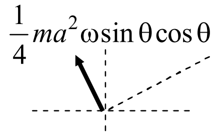
$$L(\text{due to } \omega\cos^2\theta) = \left[\frac{1}{2}ma^2 + \frac{1}{2}4m(2a)^2\right]\omega\cos^2\theta.$$

For computing the angular momentum due to $\omega\sin\theta\cos\theta$, we will use the fundamental theorem $\vec{L} = \vec{L}_{\text{motion of } cm} + \vec{L}_{\text{about } cm}$, to the discs separately and then combine the results. For the small disc,

$$\vec{L}_{\text{motion of } C_1} = m \times l \times l\omega\sin\theta\cos\theta = ml^2\omega\sin\theta\cos\theta$$



$$L_{\text{about } cm} = \frac{1}{4}ma^2\omega\sin\theta\cos\theta$$



We can do the similar calculations for the bigger disc.

Thus $L(\text{due to } \omega\sin\theta\cos\theta) = \left[\frac{1}{2}ma^2 + ml^2 + \frac{1}{2}4m(2a)^2 + 4m(2l)^2\right]\omega\sin\theta\cos\theta).$

Look at the final result carefully. By parallel axes theorem we get $I\omega\sin\theta\cos\theta$, where I is about an axis through O and parallel to $\omega\sin\theta\cos\theta$.

***Moment of inertia of a thin uniform disc (mass *m*, radius *a*) about an axis passing through its centre and inclined at an angle θ to its plane:**

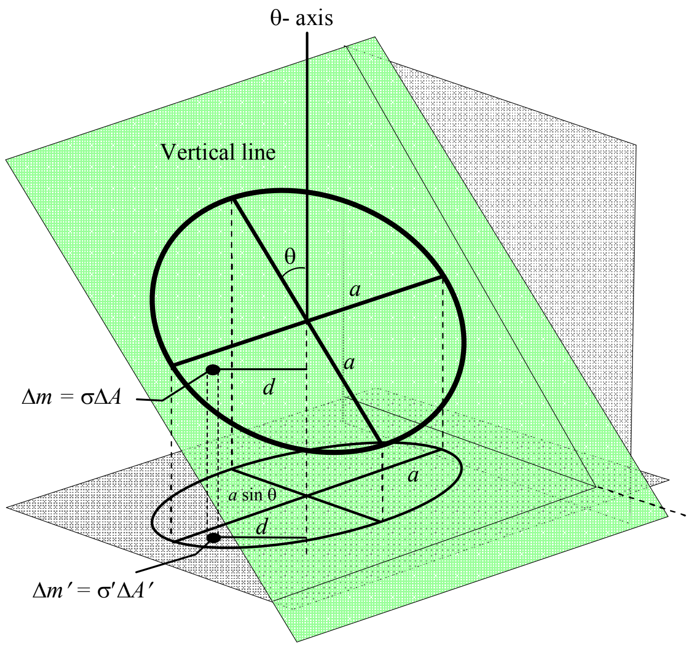
You can calculate the moment of inertia of the disc about this axis by simply adding the moments of inertia of the rings. So the problem reduces to calculating the moment of inertia of a ring about an axis through its centre and inclined to its plane.

Consider a ring of radius r and mass m. Let the centre of the ring be denoted by O. Picturise the axis-passing through its centre and at angle θ to its plane. Take an element $rd\phi$ at a convenient ϕ. Denote the position of the element by P. Mass of the element is $\lambda rd\phi$. Drop a perpendicular from the element on the θ-axis. Let the perpendicular meet the axis at point P'. The lines OP' and PP' are perpendicular. You can write $\overline{OP'}$ and $\overline{PP'}$ in terms of coordinates of points O, P and P'. Now, $\overline{OP'}\cdot\overline{PP'}=0$ gives the coordinates of point P'. The distance PP' can now be calculated. Moment of inertia of the element about the chosen axis, is $dI=(\lambda Rd\phi)(PP')^2$. Integration over the whole ring gives I. Then you can use this I for computing the moment of inertia of the a disc about the tilted axis. Do it. You will get $\frac{1}{4}ma^2(1+\sin^2\theta)$.

Alternatively, you can use a lateral idea to arrive at the answer if you know the moment of inertia of an elliptical plate (say, whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$) about its axis ($z$- axis through origin): $I=\frac{1}{4}m(a^2+b^2)$. If you don't remember this result, you can easily derive it. Take a suitable strip. Find dI and then integrate. †

We want to find out the moment of inertia of a disc (mass m, radius a) about an axis that passes through its centre and is inclined at an angle θ to its plane. Let this axis be vertical. The disc is inclined at an angle 90°– θ with the horizontal. Moment of inertia of a mass element $\Delta m=\sigma\Delta A$ about the vertical axis through the centre is $dI=(\sigma\Delta A)d^2$.

Suppose each of its mass elements $\Delta m=\sigma\Delta A$ is projected on a horizontal surface. The whole disc gives an elliptical plate. Projection of ΔA is $\Delta A'$, $\Delta A'=\Delta A\cos\theta$. Let the mass contained in area $\Delta A'$ be same as the mass contained in the area ΔA, $\Delta m'=\Delta m$ which gives $m'=m$. Mass of the elliptical plate is same as that of the disc, but the surface mass density is different.



Since $\Delta m'=\Delta m$, we have

$$\sigma'\Delta A'=\sigma\Delta A$$

or $$\sigma'\Delta A\cos\theta=\sigma\Delta A$$

or $$\sigma'=\frac{\sigma}{\cos\theta}.$$

The distance of $\Delta m'$ from the vertical axis is same as distance of Δm from it. Thus $\sum\Delta md^2=\sum\Delta m'd^2=$ moment of inertia of the elliptical plate about its axis.

The major and minor axes of the ellipse are a and $a\sin\theta$. The moment of inertia of the elliptical plate about its axis is '

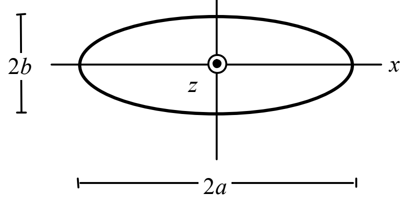
$I=\frac{m}{4}(a^2+b^2)=\frac{m}{4}(a^2+a^2\sin^2\theta)=\frac{1}{4}ma^2(1+\sin^2\theta)$, which is also the moment of inertia of the inclined disc about the axis passing through its centre and inclined at angle θ to its plane.

† We used an elliptical plate to find the moment of inertia of a disc about a certain axis. Interestingly, we can use the same disc to find the moment of inertia of the elliptical plate to begin with. That too without integration! First trick the trick itself.

Let the moment of inertia of the elliptical plate about y- axis be $I_y = kma^2$. Do you find it unreasonable to assume so? You shouldn't. By symmetry, $I_x = kmb^2$. Do you expect a different coefficient? By perpendicular axes theorem $I_z = I_x + I_y = kmb^2 + kma^2 = k(a^2 + b^2)$. Now, if $a = b$, the elliptical plate becomes a disc of $I_z = \frac{1}{2}ma^2$. Therefore,

$$km(a^2 + a^2) = \frac{1}{2}ma^2 \text{ or } k = \frac{1}{4}.$$

Hence, for the elliptical plate $I_z = \frac{1}{4}m(a^2 + b^2)$.



Brute force methods do wonders at times. Here is frontal-attack.

Imagine an x'- axis in x-z plane, passing through O and at an angle θ with x- axis. Figure 12 shows a disc, and x-, y- axes in its plane and z- axis through its centre and normal to its plane. Consider a particle of mass m_i at a distance r from the centre (point P in the figure). Drop perpendicular from m_i on x'- axis. Let this perpendicular distance be r_i. The perpendicular drawn from m_i on x'- axis meets it at P'. Let the distance of P' from O be d. Point S in the figure is foot of perpendicular drawn from P' on x- axis. (Draw a circle on a sheet. Draw x- and y- axes. Take m_i at P. Put a pencil along Ox'. Put another pencil at P such that it is perpendicular to Ox'. The pencils meet at P'. $OP' = d$. Point S on the disc is foot of perpendicular from P'. We don't use pencils for writing and drawing only; we use them for 3-D visualization also!)

In $\Delta PP'O$, $\angle PP'O = 90^\circ$, and

$$r_i^2 + d^2 = r^2. \qquad \ldots\text{(i)}$$

From $\Delta PSP'$,

$$(d\sin\theta) + (PS)^2 = r_i^2$$

or $$(d\sin\theta)^2 + [(r\sin\beta)^2 + (r\cos\beta - d\cos\theta)^2] = r_i^2$$

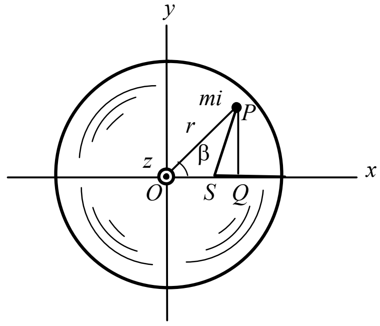
or $$d^2\sin^2\theta + r^2\sin^2\beta + r^2\cos^2\beta + d^2\cos^2\theta - 2rd\cos\beta\cos\theta = r_i^2$$

or $$d^2 + r^2 - 2rd\cos\beta\cos\theta = r_i^2 = r^2 - d^2, \text{ (From (i))}$$

which gives $d = r\cos\beta\cos\theta$.

Substituting for d in Eq. (i)

$$r_i^2 = r^2 - r^2\cos^2\beta\cos^2\theta.$$



Axis Ox', angle θ, point P', distance d are not shown in the figure. Imagine them in your head.

Fig. 12

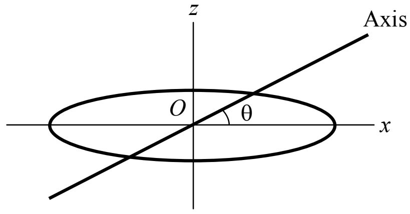
Thus $$I_\theta = \sum m_i r_i^2 = \sum m_i r^2 - \sum m_i r^2\cos^2\beta\cos^2\theta$$

$$= \frac{1}{2}ma^2 - \frac{1}{4}ma^2\cos^2\theta = \frac{1}{4}ma^2(1 + \sin^2\theta).$$

(Clearly, $\sum m_i r^2 =$ moment of inertia of the disc about its axis, which is equal to $\frac{1}{2}ma^2$ and $\sum m_i r^2 \cos^2\beta = \sum m_i (r_i \cos\beta)^2 = \sum m_i x_i^2 =$ moment of inertia of the disc about y- axis, which is $\frac{1}{4}ma^2$.)

After this frontal - attack, let us go (borrowing from chess parlance) for a counter attack, which is even more startling. Look at this.

An interesting, non-integral method to find I of a disc about an axis passing through its centre and inclined at θ to plane:



Let's give an 'ω' to the disc about axis, $\vec{\omega} = \omega\cos\theta\hat{i} + \omega\sin\theta\hat{k}$. We now know all the story of $\vec{L}$, so

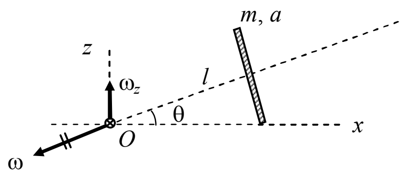
$$\vec{L}_O = I_x\omega\cos\theta\hat{i} + I_z\omega\sin\theta\hat{k} = \frac{ma^2}{4}\omega\cos\theta\hat{i} + \frac{ma^2}{2}\omega\sin\theta\hat{k}$$

$$\left|\vec{L}_{axis}\right| = L_x\cos\theta + L_z\sin\theta = \frac{ma^2}{4}\omega\cos^2\theta + \frac{ma^2}{2}\omega\sin^2\theta = \frac{1}{4}ma^2\omega\left[1+\sin^2\theta\right].$$

But $\left|\vec{L}_{axis}\right| = L_{axis}\,\omega$, therefore

$$I_{axis} = \frac{1}{4}ma^2(1+\sin^2\theta).$$

****Angular momentum of the small disc *O* due to the rotations $\vec{\omega}$ and $\vec{\omega}_z$ - brute force method.**



Finally, let's have some fun, Let's compute the angular momentum of the disc from the formula $\vec{L} = m[\vec{r}\times(\vec{\omega}\times\vec{r})]$ directly. Geometry is a little complicated. Vectors resulting from cross products may be frightening. Let's see how we manage.

Consider a particle of mass m_i at a distance of r_i from the centre of the disc. The radial line from the centre to the particle is at an angle β with the horizontal diameter (which is parallel to y- axis) of the disc, Angular velocity of the particle as seen from O is

$$\vec{\omega}_{net} = \vec{\omega} + \vec{\omega}_z.$$

Position vector of the particle m_i from O is $\vec{R} = \vec{l} + \vec{r}_i$.

The angular momentum of the disc about point O is

$$\vec{L}_O = \sum m_i[\vec{R}\times(\vec{\omega}_{net}\times\vec{R})]$$
$$= \sum m_i[(\vec{R}\cdot\vec{R})\vec{\omega}_{net} - (\vec{\omega}_{net}\cdot\vec{R})\vec{R}]$$

We have $\vec{R}\cdot\vec{R} = (\vec{l}+\vec{r}_i)\cdot(\vec{l}+\vec{r}_i) = \vec{l}\cdot\vec{l} + \vec{l}\cdot\vec{r}_i + \vec{r}_i\cdot\vec{l} + \vec{r}_i\cdot\vec{r}_i = l^2 + r_i^2$

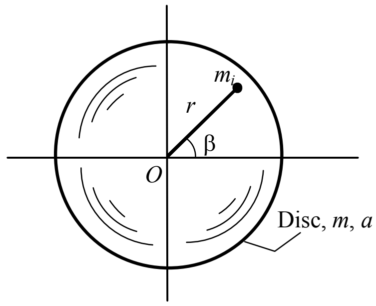
$$\vec{R}\cdot\vec{\omega}_{net} = (\vec{l}+\vec{r}_i)\cdot(\vec{\omega}+\vec{\omega}_z) = \vec{l}\cdot\vec{\omega} + \vec{l}\cdot\vec{\omega}_z + \vec{r}_i\cdot\vec{\omega} + \vec{r}_i\cdot\vec{\omega}_z$$
$$= (-l\omega + l\omega_z\sin\theta + 0 + r_i\omega_z\sin\beta\cos\theta) = -l\omega + (l\sin\theta + r_i\sin\beta\cos\theta)\omega_z.$$

Substituting from the above we get

$$\vec{L}_O = \sum m_i[(l^2 + r_i^2)(\vec{\omega}+\vec{\omega}_z) - (-l\omega + (l\sin\theta + r_i\sin\beta\cos\theta)\omega_z)(\vec{l}+\vec{r}_i)].$$

When you evaluate the product you will get an awfully lengthy, monstrous expression. But noting that

$$\sum m_i r_i^2 = \frac{1}{2}ma^2, \quad \sum m_i l^2 = ml^2,$$
$$\sum m_i (r_i \sin\beta)^2 = \frac{1}{4}ma^2, \quad \sum m_i (r_i \cos\beta)^2 = \frac{1}{4}ma^2,$$
$$\sum m_i\, r_i \sin\beta = 0, \quad \sum m_i\, r_i \cos\beta = 0$$
$$\sum m_i\, \vec{r} = 0, \quad \sum m_i\,(r_i \sin\beta)(r_i \cos\beta) = 0,$$



and dropping the terms in the product that would eventually vanish, retaining only those that matter, the monster can be easily tamed. We get

$$\vec{L}_O = \sum m_i[(l^2 + r_i^2)(\vec{\omega} + \vec{\omega}_z) + (l\omega - l\omega_z \sin\theta)\vec{l} - r_i \sin\beta\cos\theta\vec{r}_i]$$
$$= \sum m_i[(l^2 + r^2)(\vec{\omega} + \vec{\omega}_z) + (l^2\omega - l^2\omega_z \sin\theta)\hat{l} - r_i^2\omega_z \sin\beta\cos\theta\hat{r}_i]$$

The second term inside the bracket is

$$l^2\omega\hat{l} - l^2\omega_z \sin\theta\hat{l}$$
$$= l^2\omega\hat{l} - l^2\omega_z \sin\theta(\cos\theta\hat{i} + \sin\theta\hat{k})$$
$$= l^2\omega\hat{l} - l^2\omega_z \sin\theta\cos\theta\hat{i} - l^2\omega_z \sin^2\theta\hat{k}$$
$$= -l^2\vec{\omega} - l^2\omega_z \sin\theta\cos\theta\hat{i} - l^2 \sin^2\theta\vec{\omega}_z,$$

and the third terms is

$$-r_i^2\omega_z \sin\beta\cos\theta(\cos\beta\hat{j} - \sin\beta\sin\theta\hat{i} + \sin\beta\cos\theta\hat{k})$$
$$= r_i^2\omega_z \sin^2\beta\sin\theta\cos\theta\hat{i} - r_i^2 \sin^2\beta\cos^2\theta\vec{\omega}_z.$$

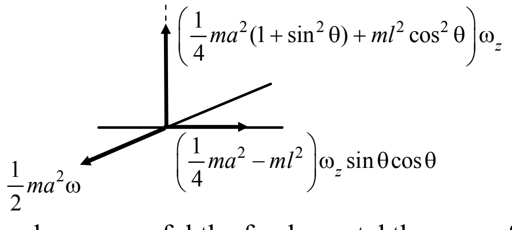
(We dropped the term $-r_i^2\omega_z \sin\beta\cos\beta\cos\theta\hat{j}$' and used $\omega_z\hat{k} = \vec{\omega}_z$.)

Thus, $\vec{L}_O = \sum m_i\left[(l^2 + r_i^2)(\vec{\omega} + \vec{\omega}_z) - l^2\vec{\omega} - l^2\omega_z \sin\theta\cos\theta\hat{i} - l^2 \sin^2\theta\vec{\omega}_z + r_i^2\omega_z \sin^2\beta\sin\theta\cos\theta\hat{i} - r_i^2 \sin^2\beta\cos^2\theta\right.$

which simplifies to

$$\vec{L}_O = \sum m_i\left[r_i^2\vec{\omega} + l^2\cos^2\theta\vec{\omega}_z + r_i^2\vec{\omega}_z + r_i^2 \sin^2\beta\cos^2\theta\vec{\omega}_z + (r_i^2 \sin^2\beta - l^2)\omega_z \sin\theta\cos\theta\hat{i}\right]$$
$$= \frac{1}{2}ma^2\vec{\omega} + ml^2\cos^2\theta\vec{\omega} + \frac{1}{2}ma^2\vec{\omega}_z + \frac{1}{4}ma^2\cos^2\vec{\omega}_z + \left(\frac{1}{4}ma^2 - ml^2\right)\omega_z \sin\theta\cos\theta\hat{i}$$
$$= \frac{1}{2}ma^2\vec{\omega} + \left[\frac{1}{4}ma^2(1 + \sin^2\theta) + ml^2\cos^2\theta\right]\vec{\omega}_z + \left(\frac{1}{4}ma^2 - ml^2\right)\omega_z \sin\theta\cos\theta\hat{i}$$

These vectors are drawn in the figure below.



This calculation makes us realize how powerful the fundamental theorem of angular momentum is!

One can ask a few interesting questions regarding the motion of the assembly.

(i) Does the angular momentum of the assembly about point O change with time or it remains constant? If it changes, what is the rate of change with respect to time?

(ii) What are the forces on the assembly? What about the direction of the resultant force?

(iii) What is the torque on the assembly about point O? About its centre of mass?

(iv) The magnitude of the resultant force on the bigger disc is n times greater than that on the small disc. Find n.

(v) What is the magnitude of the torque on the small disc about point O?

(vi) Find the normal forces on the discs by the flat surface.

(vii) Comment on the stresses that develop, of if any, in the massless rod connecting the centres of the discs.